Tishk International University/Sulaimani
Faculty of Engineering

Calculus I

TOPIC: Real Numbers

1st stage – Fall semester – 1st Lecture

Instructor:

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Real Numbers

- **Real numbers** are used in everyday life to describe quantities such as age, miles per gallon, and population. Real numbers are represented by symbols such as

$$-5, 9, 0, \frac{4}{3}, 0.666\ .\ .\ .\ ,\ 28.21,\ \sqrt{2},\ \pi, \text{ and } \sqrt[3]{-32}$$

- Here are some important **subsets**

$\{1, 2, 3, 4, \ldots\}$	Set of natural numbers
$\{0, 1, 2, 3, 4, \ldots\}$	Set of whole numbers
$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	Set of integers

A real number is **rational** if it can be written as the ratio p/q of two integers, where $q \neq 0$. For instance, the numbers

$$\frac{1}{3} = 0.3333\ldots = 0.\overline{3},\ \frac{1}{8} = 0.125, \text{ and } \frac{125}{111} = 1.126126\ldots = 1.\overline{126}$$

A real number that cannot be written as the ratio of two integers is called **irrational.**
Irrational numbers have infinite nonrepeating decimal representations.
For instance, the numbers are irrational. (The symbol ≈ means "is approximately equal to.")

$$\sqrt{2} = 1.4142135\ldots \approx 1.41 \quad \text{and} \quad \pi = 3.1415926\ldots \approx 3.14$$

Next Figure shows subsets of real numbers and their relationships to each other.

- Real numbers
  - Irrational numbers: $-\sqrt{5},\ \sqrt{3},\ \pi$
  - Rational numbers: $-\frac{1}{2},\ 0,\ \frac{2}{3}$
    - Noninteger fractions (positive and negative): $-\frac{2}{7},\ \frac{4}{3},\ 0.5$
    - Integers: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$
      - Negative integers: $\ldots, -3, -2, -1$
      - Whole numbers: $0, 1, 2, \ldots$
        - Natural numbers: $1, 2, 3, \ldots$
        - Zero

In *decimal form,* you can recognize rational numbers as decimals that terminate

$$\tfrac{1}{2} = 0.5 \quad \text{or} \quad \tfrac{3}{8} = 0.375$$

or repeat

$$\tfrac{4}{3} = 1.\overline{3} \quad \text{or} \quad \tfrac{2}{11} = 0.\overline{18}.$$

Irrational numbers are represented by decimals that neither terminate nor repeat, as in

$$\sqrt{2} = 1.414213562\ldots$$

or

$$\pi = 3.141592653\ldots.$$

Example: Classifying Real Numbers

Determine which numbers in the set

$$\left\{-13, -\sqrt{5}, -1, -\frac{1}{3}, 0, \frac{5}{8}, \sqrt{2}, \pi, 7\right\}$$

are (a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

Solution

a. Natural numbers: $\{7\}$

b. Whole numbers: $\{0, 7\}$

c. Integers: $\{-13, -1, 0, 7\}$

d. Rational numbers: $\left\{-13, -1, -\frac{1}{3}, 0, \frac{5}{8}, 7\right\}$

e. Irrational numbers: $\{-\sqrt{5}, \sqrt{2}, \pi\}$

- **Example:**
- Which of the numbers in the following set are (a) natural numbers, (b) integers, (c) rational numbers, and (d) irrational numbers?

$$\left\{\frac{1}{2},\ -1,\ 0,\ 4,\ -\frac{5}{8},\ \frac{4}{2},\ -\frac{3}{1},\ 0.86,\ \sqrt{2},\ \sqrt{9}\right\}$$

- Class Work: Determine which numbers in the set are

(a) natural numbers, (b) whole numbers, (c) integers, (d) rational numbers, and (e) irrational numbers.

1. $\left\{-9, -\frac{7}{2}, 5, \frac{2}{3}, \sqrt{2}, 0, 1, -4, 2, -11\right\}$

2. $\left\{\sqrt{5}, -7, -\frac{7}{3}, 0, 3.12, \frac{5}{4}, -3, 12, 5\right\}$

3. $\{2.01, 0.666\ .\ .\ .\ , -13, 0.010110111\ .\ .\ .\ , 1, -6\}$

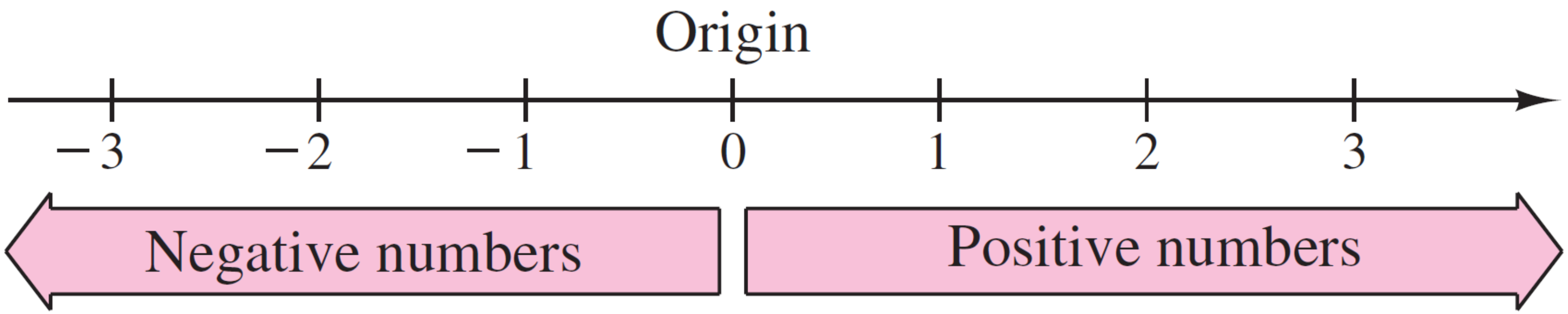
4. $\{2.3030030003\ .\ .\ .\ , 0.7575, -4.63, \sqrt{10}, -75, 4\}$

5. $\left\{-\pi, -\frac{1}{3}, \frac{6}{3}, \frac{1}{2}\sqrt{2}, -7.5, -1, 8, -22\right\}$

6. $\left\{25, -17, -\frac{12}{5}, \sqrt{9}, 3.12, \frac{1}{2}\pi, 7, -11.1, 13\right\}$

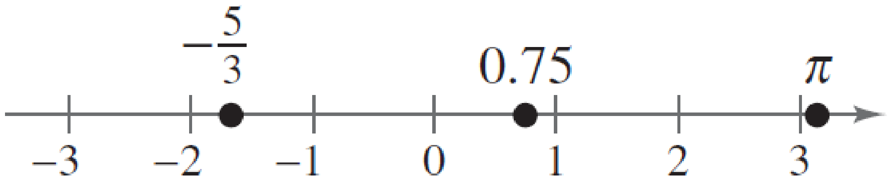
The Real Number Line

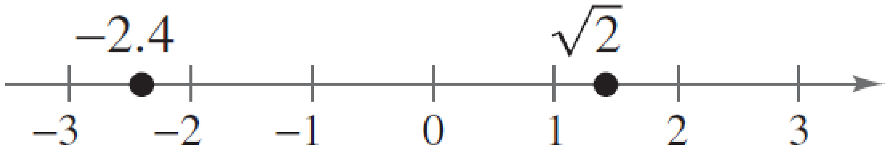
- The diagram used to represent the real numbers is called the **real number line.** It consists of a horizontal line with a point (the **origin**) labeled 0.
- Numbers to the left of 0 are **negative** and numbers to the right of 0 are **positive**, as shown in Figure.
- The real number zero is neither positive nor negative. So, the term **nonnegative** implies that a number may be positive or zero.



Plotting Real Numbers

- Drawing the point on the real number line that corresponds to a real number is called **plotting** the real number.
- *Each point on the real number line corresponds to exactly one real number, and each real number corresponds to exactly one point on the real number line.*
- *Example:*





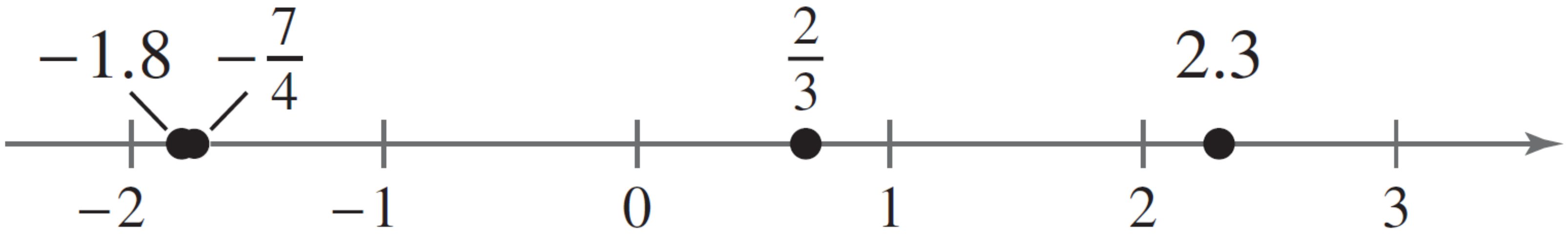
- Example: Plot the real numbers on the real number line:

a. $-\frac{7}{4}$

b. 2.3

c. $\frac{2}{3}$

d. -1.8



a. The point representing the real number $-\frac{7}{4} = -1.75$ lies between -2 and -1, but closer to -2, on the real number line.

b. The point representing the real number 2.3 lies between 2 and 3, but closer to 2, on the real number line.

c. The point representing the real number $\frac{2}{3} = 0.666\ldots$ lies between 0 and 1, but closer to 1, on the real number line.

d. The point representing the real number -1.8 lies between -2 and -1, but closer to -2, on the real number line. Note that the point representing -1.8 lies slightly to the left of the point representing $-\frac{7}{4}$.

Example:

- plot the real numbers on the real number line.

(a) 3 (b) $\frac{7}{2}$ (c) $-\frac{5}{2}$ (d) -5.2

- **Class work:**
- plot the real numbers on the real number line.

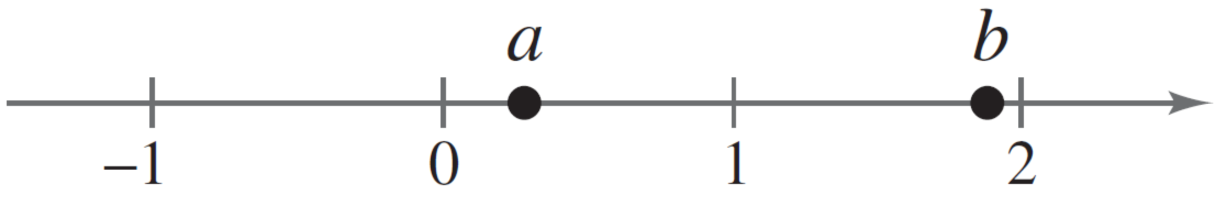
(a) 8.5 (b) $\frac{4}{3}$ (c) -4.75 (d) $-\frac{8}{3}$

Ordering Real Numbers

Definition of Order on the Real Number Line

If a and b are real numbers, a is less than b if $b - a$ is positive. The **order** of a and b is denoted by the **inequality** $a < b$. This relationship can also be described by saying that b is *greater than* a and writing $b > a$. The inequality $a \leq b$ means that a is *less than or equal to* b, and the inequality $b \geq a$ means that b is *greater than or equal to* a. The symbols $<$, $>$, $\leq$, and $\geq$ are *inequality symbols.*

Geometrically, this definition implies that $a < b$ if and only if a lies to the *left* of b on the real number line



Example:

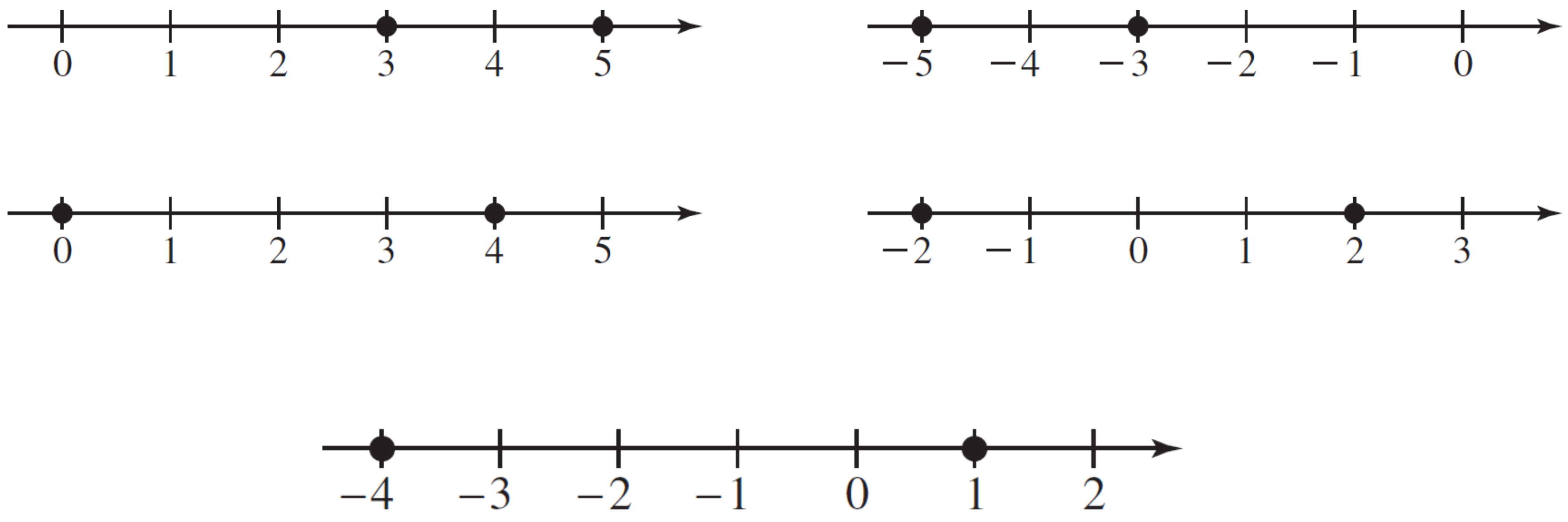
a. $3 < 5$, because 3 lies to the *left* of 5.

b. $-3 > -5$, because -3 lies to the *right* of -5.

c. $4 > 0$, because 4 lies to the *right* of 0.

d. $-2 < 2$, because -2 lies to the *left* of 2.

e. $1 > -4$, because 1 lies to the *right* of -4.



Intervals on the Real Number Line

- There are two types:
- **Open interval:**
- If $a < b$, then the **open interval** from a to b consists

of all numbers between a and b and is denoted (a, b).

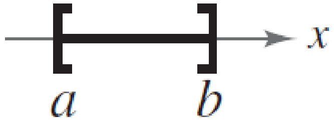
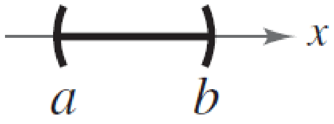
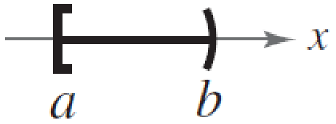
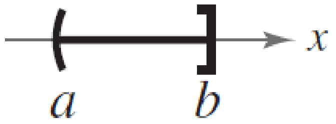
- Endpoints are **excluded** from the interval
- **Closed interval:**
- from a to b **includes the endpoints** and

is denoted [a, b].

- Using set-builder notation, we can write

$$(a, b) = \{x \mid a < x < b\} \qquad [a, b] = \{x \mid a \le x \le b\}$$

Bounded Intervals on the Real Number Line

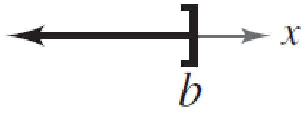
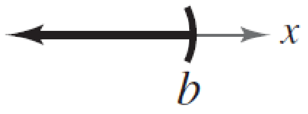
Notation	*Interval Type*	*Inequality*	*Graph*
$[a, b]$	Closed	$a \leq x \leq b$	
(a, b)	Open	$a < x < b$	
$[a, b)$		$a \leq x < b$	
$(a, b]$		$a < x \leq b$	

Unbounded Intervals on the Real Number Line

The symbols ∞, **positive infinity,** and $-\infty$, **negative infinity,** do not represent real numbers. They are simply convenient symbols used to describe the unboundedness of an interval such as $(1, \infty)$ or $(-\infty, 3]$.

WARNING / CAUTION

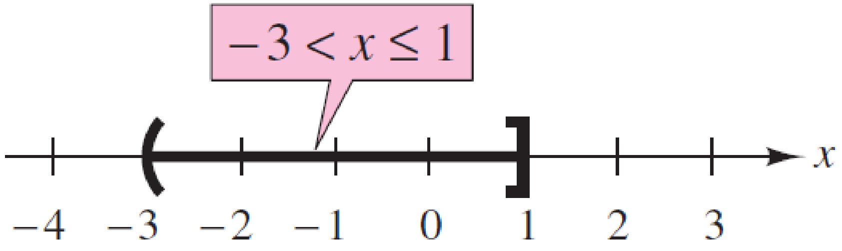
Whenever you write an interval containing ∞ or $-\infty$, always use a parenthesis and never a bracket. This is because ∞ and $-\infty$ are never an endpoint of an interval and therefore are not included in the interval.

Notation	*Interval Type*	*Inequality*	*Graph*
$[a, \infty)$		$x \geq a$	
(a, ∞)	Open	$x > a$	
$(-\infty, b]$		$x \leq b$	
$(-\infty, b)$	Open	$x < b$	
$(-\infty, \infty)$	Entire real line	$-\infty < x < \infty$	

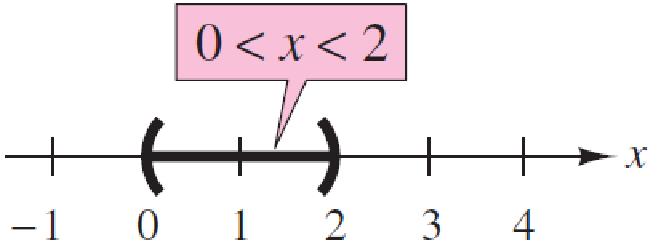
Graphs of Inequalities

- **Example**:
- **Sketch the graph of each inequality.**

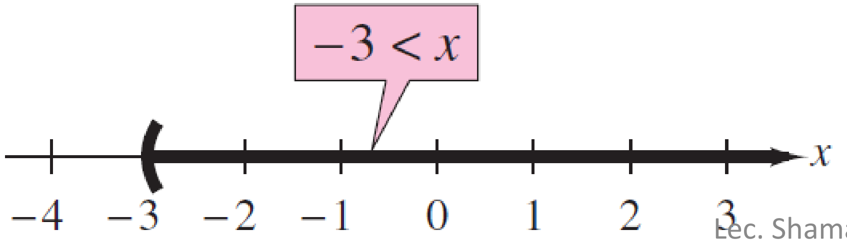
a. The graph of $-3 < x \leq 1$ is a bounded interval.



b. The graph of $0 < x < 2$ is a bounded interval.



c. The graph of $-3 < x$ is an unbounded interval.



d. The graph of $x \leq 2$ is an unbounded interval.